SADLER MATHEMATICS SPECIALIST UNIT 2

WORKED SOLUTIONS

Chapter 12 Proof

Exercise 12A

Question 1

If we square any even counting number greater than 2 and the subtract 1, we get a multiple of 5.

$4^2 - 1 = 15$
$6^2 - 1 = 35$
$8^2 - 1 = 63$

Conjecture is false as shown by last example.

Question 2

The cube of any even integer is always a multiple of 8.

$2^3 = 8$
$4^3 = 64 = 8 \times 8$
$6^3 = 216 = 27 \times 8$

3 examples appear to support conjecture.

Let an even integer be represented by $2n$, where $n \in \mathbb{Z}$.
$(2n)^3 = 8n^3$ which is a multiple of three.
Conjecture is true.

Question 3

All multiples of 5 area also multiples of 10.

Conjecture is false as 15 is a multiple of 5 but not a multiple of 10.

Question 4

All right triangles are isosceles.

Conjecture is false as a 3,4 5 triangle is a right angled scalene triangle.

Question 5

If we add together an integer squared, six times the integer and 9 we get a square number.

$$3^2+6(3)+9=36=6^2$$
$$4^2+6(4)+9=49=7^2$$
$$10^2+6(10)+9=169=13^2$$

3 examples appear to support conjecture.

Let n represent an integer, i.e.n $\in \mathbb{Z}$.

$$n^2+6n+9=(n+3)^2$$

Conjecture is true.

Question 6

The sum of three consecutive positive integers will always be a multiple of 3.

$$3+4+5=12=4\times3$$
$$5+6+7=18=6\times3$$
$$10+11+12=33=11\times3$$

Let the first number be n, $n\in\mathbb{Z}$.

Three consective numbers would be represented by n, $n+1$ and $n+2$.

$$n+n+1+n+2=3n+3=3(n+1)$$

$3(n+1)$ is a multiple of three.

Conjecture is true.

Question 7

The product of two even numbers is always even.

$4 \times 6 = 24$
$6 \times 10 = 60$
$8 \times 10 = 80$

3 examples appear to support conjecture.

Let $n, m \in \mathbb{Z}$.
$2n$ and $2m$ represent even numbers.
$2n \times 2m = 4mn = 2(2mn)$
$2(2mn)$ is a multiple of 2 and therefore even.
Conjecture is true.

Question 8

The square of an odd number is always an odd number.

$5^2 = 25$
$7^2 = 49$
$11^2 = 121$

3 examples appear to support conjecture.

Let $2n+1$ represent an odd number, $n \in \mathbb{Z}$.
$(2n+1)^2 = 4n^2 + 4n + 1 = 2(2n^2 + 2n) + 1.$
$(2n^2 + 2n) \in \mathbb{Z}$ as $n \in \mathbb{Z}$.
$2(2n^2 + 2n) + 1$ is odd as 2 multiplied by any integer add 1 is odd.
Conjecture is true.

Question 9

The product of two consecutive even whole numbers is always a multiple of 8.

$4\times 6=24=3\times 8$
$10\times 12=120=15\times 8$
$12\times 14=168=21\times 8$

3 examples appear to support conjecture.

Consider consecutive even numbers, $2n\,\&\,2(n+1)$.
$2n\times 2(n+1)=4n\left(n+1\right)$

If n is even, $n=2k, k\in\mathbb{Z}$.
$4n\left(n+1\right)$
$=4\left(2k\right)\left(2k+1\right)$
$=8k\left(2k+1\right)$ which is a multiple of 8.

f n isodd, $n=2k+1, k\in\mathbb{Z}$.
$4n\left(n+1\right)$
$=4\left(2k+1\right)\left(2k+1+1\right)$
$=4\left(2k+1\right)\left(2k+2\right)$
$=4\left(2k+1\right)2\left(k+1\right)$
$=8\left(k+1\right)\left(2k+1\right)$ which is a multiple of 8.
Conjecture is true.

Question 10

Multiplying any odd counting number by itself and then adding 7 always gives a multiple of 8.

$5^2 + 7 = 32 = 4 \times 8$

$3^2 + 7 = 16 = 2 \times 8$

$9^2 + 7 = 88 = 11 \times 8$

3 examples appear to support conjecture.

Let $2n+1$ represent any odd counting number, $n \in \mathbb{Z}, n \geq 0$.

$(2n+1)^2 + 7$

$= 4n^2 + 4n + 8$

$= 4n(n+1) + 8$

If n is even, $n = 2k, k \in \mathbb{Z}, k \geq 0$.

$4n(n+1) + 8$

$= 4(2k)(2k+1) + 8$

$= 8[(k)(2k+1) + 1]$ which is a multiple of 8.

If n is odd, $n = 2k+1, k \in \mathbb{Z}, k \geq 0$.

$4n(n+1) + 8$

$= 4(2k+1)(2k+1+1) + 8$

$= 4(2k+1)(2k+2) + 8$

$= 8[(k+1)(2k+3) + 1]$ which is a multiple of 8.

Conjecture is true.

Question 11

a Let $x = 0.555\dot{5}$

$10x = 5.555\dot{5}$

$9x = 5$

$x = \frac{5}{9}$

b Let $x = 0.75\overline{75}$

$100x = 75.75\overline{75}$

$99x = 75$

$x = \frac{75}{99} = \frac{25}{33}$

c Let $x = 0.6363\overline{63}$

$100x = 63.63\overline{63}$

$99x = 63$

$x = \frac{63}{99} = \frac{7}{11}$

d Let $x = 2.231\overline{231}$

$1000x = 2231.\overline{231}$

$999x = 2229$

$x = \frac{2229}{999} = \frac{743}{333}$

e Let $x = 0.23144\dot{4}$

$1000x = 231.44\dot{4}$

$10000x = 2314.44\dot{4}$

$9000x = 2083$

$x = \frac{2083}{9000}$

Question 12

Assume that $\sqrt{2}$ is rational

i.e. $\sqrt{2}$ can be expressed in the form $\frac{a}{b}$, $a, b \in \mathbb{Z}, b \neq 0$ with *a* and *b* being co-prime.

$$\sqrt{2} = \frac{a}{b}$$
$$2 = \frac{a^2}{b^2}$$
$$2b^2 = a^2$$

a^2 is even therefore a is even.

We can then write $a = 2k, k \in \mathbb{Z}$

$$a^2 = (2k)^2 = 2b^2$$
$$4k^2 = 2b^2$$
$$2k^2 = b^2$$

b^2 is even therefore b is even.

However, if *a* and *b* are both even, they have a common factor of 2 and we have a contradiction to our initial assumption that *a* and *b* are co-prime.

Our assumption that $\sqrt{2}$ is rational must be false, hence $\sqrt{2}$ is irrational.

Exercise 12B

Question 1

If the number is even, we can represent it as $2n, n \in \mathbb{Z}$.

$(2n)^2 = 4n^2$

$4n^2$ is a multiple of 4 and also even.

If the number is odd, we can represent it as $2n+1, n \in Z$. ,

$(2n+1)^2 = 4n^2 + 4n + 1$

$= 4(n^2 + n) + 1$

$4(n^2 + n) + 1$ is one more than a multiple of 4 and therefore odd.

Question 2

There are five possibilities.

The integer chosen is either a multiple of 5, 1 more than a multiple of 5, 2 more than a multiple of 5, 3 more than a multiple of 5 or four more than a multiple of 5.

If the number chosen is a multiple of 5, we can write $x = 5n, n \in \mathbb{Z}$.

$$\begin{aligned}(5n)^2 &= 25n^2 \\ &= 5(5n^2)\end{aligned}$$

The result is a multiple of 5.

If the number chosen is one more than a multiple of 5, we can write $x = 5n+1, n \in \mathbb{Z}$.

$$\begin{aligned}(5n+1)^2 &= 25n^2 + 10n + 1 \\ &= 5(5n^2 + 2n) + 1\end{aligned}$$

The result is one more than a multiple of 5.

If the number chosen is two more than a multiple of 5, we can write $x = 5n+2, n \in \mathbb{Z}$.

$$\begin{aligned}(5n+2)^2 &= 25n^2 + 20n + 4 \\ &= 5(5n^2 + 4n) + 4\end{aligned}$$

The result is four more than a multiple of 5.

If the number chosen is three more than a multiple of 5, we can write $x = 5n+3, n \in \mathbb{Z}$.

$$\begin{aligned}(5n+3)^2 &= 25n^2 + 30n + 9 \\ &= 5(5n^2 + 6n + 1) + 4\end{aligned}$$

The result is four more than a multiple of 5.

If the number chosen is four more than a multiple of 5, we can write $x = 5n+4, n \in \mathbb{Z}$.

$$\begin{aligned}(5n+4)^2 &= 25n^2 + 40n + 16 \\ &= 5(5n^2 + 8n + 3) + 1\end{aligned}$$

The result is one more than a multiple of 5.

Question 3

There are three possibilities.

The integer chosen is either a multiple of 3, 1 more than a multiple of 3 or two more than a multiple of 5.

If the number chosen is a multiple of 3, we can write $x = 3n, n \in \mathbb{Z}$.

$$\begin{aligned}(3n)^3 &= 27n^3 \\ &= 9(3n^3)\end{aligned}$$

The result is a multiple of 9.

If the number chosen is one more than a multiple of 3, we can write $x = 3n+1, n \in \mathbb{Z}$.

$$\begin{aligned}(3n+1)^3 &= 27n^3 + 27n^2 + 9n + 1 \text{ (Use CP to expand)} \\ &= 9(3n^3 + 3n^2 + n) + 1\end{aligned}$$

The result is one more than a multiple of 9.

If the number chosen is two more than a multiple of 3, we can write $x = 3n+2, n \in \mathbb{Z}$.

$$\begin{aligned}(3n+2)^3 &= 27n^3 + 54n^2 + 36n + 8 \text{ (Use CP to expand)} \\ &= 9(3n^3 + 6n^2 + 4n + 1) - 1\end{aligned}$$

The result is one less than a multiple of 9.

Question 4

There are two possibilities, the term is either even or odd.

If the term is even, we can write

$$\begin{aligned} T_n &= 2k, k \in \mathbb{Z}^+. \\ T_{n+1} &= 3T_n + 2 \\ &= 3(2k) + 2 \\ &= 6k + 2 \\ &= 2(3k + 1) \end{aligned}$$

T_{n+1} is also even.

If the term is odd, we can write

$$\begin{aligned} T_n &= 2k + 1, k \in \mathbb{Z}^+. \\ T_{n+1} &= 3T_n + 2 \\ &= 3(2k + 1) + 2 \\ &= 6k + 5 \\ &= 2(3k + 2) + 1 \end{aligned}$$

T_{n+1} is also odd.

For this sequence, the next term will have the same parity as the term it follows.

Question 5

There are five possibilities. The integer chosen, *x*, is either a multiple of 5, 1 more than a multiple of 5, 2 more than a multiple of 5, 3 more than a multiple of 5 or four more than a multiple of 5.

$$\left(x^5 - x\right) = x\left(x-1\right)\left(x+1\right)\left(x^2+1\right)$$

As long as one of the factors is a multiple of 5, the product is also a multiple of 5.

If *x* is a multiple of 5, then we can write

$x = 5n, n \in \mathbb{Z}.$

$\left(x^5 - x\right)$

$= x\left(x-1\right)\left(x+1\right)\left(x^2+1\right)$

$= 5n\left(5n-1\right)\left(5n+1\right)\left(\left(5n\right)^2+1\right)$ which is a multiple of 5.

If *x* is one more than a multiple of 5, then we can write

$x = 5n+1, n \in \mathbb{Z}.$

$\left(x^5 - x\right)$

$= x\left(x-1\right)\left(x+1\right)\left(x^2+1\right)$

$= (5n+1)\left(5n+1-1\right)\left(5n+1+1\right)\left(\left(5n+1\right)^2+1\right)$

$= (5n+1)\left(5n\right)\left(5n+2\right)\left(\left(5n+1\right)^2+1\right)$

$=5n(5n+1)\left(5n+2\right)\left(\left(5n+1\right)^2+1\right)$ which is a multiple of 5.

If *x* is two more than a multiple of 5, then we can write

$x = 5n+2, n \in \mathbb{Z}.$

$\left(x^5 - x\right)$

$= x\left(x-1\right)\left(x+1\right)\left(x^2+1\right)$

$= (5n+2)\left(5n+2-1\right)\left(5n+2+1\right)\left(\left(5n+2\right)^2+1\right)$

$= (5n+2)\left(5n+1\right)\left(5n+3\right)\left(25n^2+20n+4+1\right)$

$= (5n+2)\left(5n+1\right)\left(5n+3\right)\left(25n^2+20n+5\right)$

$=5(5n^2+4n+1)\ (5n+2)\left(5n+1\right)\left(5n+3\right)$ which is a multiple of 5.

If x is three more than a multiple of 5, then we can write

$x = 5n + 3, n \in \mathbb{Z}.$

$\left(x^5 - x\right)$

$= x\left(x-1\right)\left(x+1\right)\left(x^2+1\right)$

$= (5n+3)\left(5n+3-1\right)\left(5n+3+1\right)\left(\left(5n+3\right)^2+1\right)$

$= (5n+3)\left(5n+2\right)\left(5n+4\right)\left(25n^2+30n+9+1\right)$

$= (5n+2)\left(5n+1\right)\left(5n+3\right)\left(25n^2+30n+10\right)$

$=5(5\text{n}^2+6n+2)\ (5n+2)\left(5n+1\right)\left(5n+3\right)$ which is a multiple of 5.

If x is four more than a multiple of 5, then we can write

$x = 5n + 4, n \in \mathbb{Z}.$

$\left(x^5 - x\right)$

$= x\left(x-1\right)\left(x+1\right)\left(x^2+1\right)$

$= (5n+4)\left(5n+4-1\right)\left(5n+4+1\right)\left(\left(5n+4\right)^2+1\right)$

$= (5n+4)\left(5n+3\right)\left(5n+5\right)\left(\left(5n+4\right)^2+1\right)$

$= 5(\text{n}+1)(5n+4)\left(5n+3\right)\left(\left(5n+4\right)^2+1\right)$ which is a multiple of 5.

For integer $x > 1, x^5 - x$ is always a multiple of 5.

If x is even, then $x^5 - x$ has a factor of 5 and a factor of 2 and is therefore a multiple of 10.

$x, x+1$ and $x+2$ are consecutive numbers.

If x is odd, $x+1$ is even and again, $x^5 - x$ has a factor of 5 and a factor of 2 and is therefore a multiple of 10.

To be a multiple of 20 we require a multiple of 5 and two even numbers as factors $(20 = 2 \times 2 \times 5)$

If x is even, then all the other factors of $x^5 - x$ are odd. We only have one even factor and therefore $x^5 - x$ cannot be a multiple of 20.

If x is odd, then $x-1$ and $x+1$ are even. We then have two even factors and a multiple of 5 and therefore $x^5 - x$ is a multiple of 20.

Question 6

There are seven possibilities. The integer chosen, x, is either a multiple of 7, 1 more than a multiple of 7, 2 more than a multiple of 7, 3 more than a multiple of 7, four more than a multiple of 7, five more than a multiple of 7 or 6 more than a multiple of 7.

$$\left(x^7 - x\right) = x(x-1)(x+1)\left(x^2 + x + 1\right)\left(x^2 - x + 1\right)$$

As long as one of the factors is a multiple of 7, the product is also a multiple of 7.

If x is a multiple of 7, we can write $x = 7n, n \in \mathbb{Z}$.

As $x = 7n$, we have a factor that is a multiple of 7, so $x^7 - x$ is a multiple of 7.

If x is one more than a multiple of 7, we can write it as $7n + 1, n \in \mathbb{Z}$.

As $x = 7n + 1$, $(x-1) = (7n + 1 - 1) = 7n$.

We have a factor that is a multiple of 7, so $x^7 - x$ is a multiple of 7.

If x is two more than a multiple of 7, we can write it $x = 7n + 2, n \in \mathbb{Z}$.

$$\begin{aligned}\text{As } x = 7n+2,\ \left(x^2 + x + 1\right) &= (7n+2)^2 + (7n+2) + 1 \\ &= 49n^2 + 35n + 7 \\ &= 7\left(7n^2 + 5n + 1\right)\end{aligned}$$

We have a factor that is a multiple of 7, so $x^7 - x$ is a multiple of 7.

If x is three more than a multiple of 7, we can write $x = 7n + 3, n \in \mathbb{Z}$.

$$\begin{aligned}\text{As } x = 7n+3, \left(x^2 - x + 1\right) &= (7n+3)^2 - (7n+3) + 1 \\ &= 49n^2 + 35n + 7 \\ &= 7\left(7n^2 + 5n + 1\right)\end{aligned}$$

We have a factor that is a multiple of 7, so $x^7 - x$ is a multiple of 7.

If x is four more than a multiple of 7, we can write it as $7n + 4, n \in \mathbb{Z}$.

$$\begin{aligned}\text{As } x = 7n+4, \left(x^2 + x + 1\right) &= (7n+4)^2 + (7n+4) + 1 \\ &= 49n^2 + 56n + 21 \\ &= 7\left(7n^2 + 8n + 3\right)\end{aligned}$$

We have a factor that is a multiple of 7, so $x^7 - x$ is a multiple of 7.

If x is five more than a multiple of 7, we can write it as $7n+5, n \in \mathbb{Z}$.

$$\text{As } x = 7n+5, \left(x^2 - x + 1\right) = \left(7n+5\right)^2 - \left(7n+5\right) + 1$$
$$= 49n^2 + 63n + 21$$
$$= 7\left(7n^2 + 9n + 3\right)$$

We have a factor that is a multiple of 7, so $x^7 - x$ is a multiple of 7.

If x is six more than a multiple of 7, we can write it as $7n+6, n \in \mathbb{Z}$.

$$\text{As } x = 7n+6, x+1 = 7n+6+1$$
$$= 7n+7$$
$$= 7(n+1)$$

We have a factor that is a multiple of 7, so $x^7 - x$ is a multiple of 7.

Hence $x^7 - x$ is always a multiple of 7 for $x > 1$.

Question 7

No John's conjecture is not correct.

$6^3 - 6 = 210$

210 is not a multiple of 12.

$$\begin{aligned} x^3 - x &= x(x^2 - 1) \\ &= x(x-1)(x+1) \end{aligned}$$

In any three consecutive integers, there is a multiple of three and one even number which means the product is always a multiple of 6.

There are three possibilities for the first integer x. The integer chosen, x, is either a multiple of 3, 1 more than a multiple of 3 or 2 more than a multiple of 3.

If x is a multiple of 3, we can write $x = 3n, n \in \mathbb{Z}$.

If $x = 3n$,

$x(x-1)(x+1) = 3n(3n-1)(3n+1)$.

We have a factor which is a multiple of three and one of the three numbers is also an even number, therefore $x(x-1)(x+1)$ is a multiple of 6.

If x is one more than a multiple of 3, we can write $x = 3n+1, n \in \mathbb{Z}$.

If $x = 3n+1$,

$$\begin{aligned} x(x-1)(x+1) &= (3n+1)(3n+1-1)(3n+1+1) \\ &= (3n+1)3n(3n+2) \end{aligned}$$

We have a factor which is a multiple of three and one of the three numbers is also an even number, therefore $x(x-1)(x+1)$ is a multiple of 6.

If x is two more than a multiple of 3, we can write $x = 3n+2, n \in \mathbb{Z}$.

If $x = 3n+2$,

$$\begin{aligned} x(x-1)(x+1) &= (3n+2)(3n+2-1)(3n+2+1) \\ &= (3n+2)(3n+1)(3n+3) \\ &= 3(n+1)(3n+1)(3n+2) \end{aligned}$$

We have a factor which is a multiple of three and one of the three numbers is also an even number, therefore $x(x-1)(x+1)$ is a multiple of 6.

Exercise 12C

Question 1

RTP: $1+2+3+4+...+n=\frac{1}{2}n\left(n+1\right), \qquad \forall n\in\mathbb{Z},\ n\geq 1.$

When n = 1:

LHS = 1

RHS$=\frac{1}{2}\times 1\times 2=1$

The initial case is true.

Assume the statement is true for $n=k$.

i.e. $1+2+3+4+...+n=\frac{1}{2}k\left(k+1\right), \qquad \forall k\in\mathbb{Z}, k\geq 1$

When $n=k+1$, RHS $=\frac{1}{2}(k+1)(k+2)$

$$\begin{aligned}\text{LHS} &= 1+2+3+4+...+k+\left(k+1\right)\\ &= \frac{1}{2}k\left(k+1\right)+\left(k+1\right)\\ &= \left(k+1\right)(\frac{1}{2}k+1)\\ &= \frac{1}{2}(k+1)(k+2)\\ &= \text{RHS}\end{aligned}$$

If the statement is true when $n=k$, it is also true for $n=k+1$.

Given that is true when n = 1, by induction, $1+2+3+4+...+n=\frac{1}{2}n\left(n+1\right), \qquad \forall n\in\mathbb{Z},\ n\geq 1.$

Question 2

RTP: $1\times 2+2\times 3+3\times 4+...+n\left(n+1\right)=\frac{n}{3}\left(n+1\right)\left(n+2\right), \quad \forall n\in \mathbb{Z},\ n\ge 1.$

When $n = 1$

$\text{LHS}: 1\times (1+1)=2$

$\text{RHS}: \frac{1}{3}\left(1+1\right)\left(1+2\right)=2$

The initial case is true.

Assume the statement is true for $n=k.$

i.e. $1\times 2+2\times 3+3\times 4+...+k\left(k+1\right)=\frac{k}{3}\left(k+1\right)\left(k+2\right), \quad \forall k\in \mathbb{Z}, k\ge 1.$

When $n=k+1,$ RHS= $\frac{k+1}{3}\left(k+2\right)\left(k+3\right)$

$$
\begin{aligned}
\text{LHS} &= 1\times 2+2\times 3+...+k\left(k+1\right)+\left(k+1\right)\left(k+2\right)\\
&=\frac{k}{3}\left(k+1\right)\left(k+2\right)+\left(k+1\right)\left(k+2\right)\\
&=\left(k+1\right)\left(k+2\right)\left(\frac{k}{3}+1\right)\\
&=\frac{1}{3}\left(k+1\right)\left(k+2\right)\left(k+3\right)\\
&=\text{RHS}
\end{aligned}
$$

If the statement is true when $n=k,$ it is also true for $n=k+1.$

Given that is true when n = 1, by induction

$1\times 2+2\times 3+3\times 4+...+n\left(n+1\right)=\frac{n}{3}\left(n+1\right)\left(n+2\right), \quad \forall n\in \mathbb{Z},\ n\ge 1.$

Question 3

RTP: $2+4+8+...+2^n = 2^{n+1}-2, \quad \forall n \in \mathbb{Z},\ n \geq 1..$

When n = 1:

LHS = 2

RHS= $2^{1+1}-2=2$

The initial case is true.

Assume the statement is true for $n=k$.

i.e. $2+4+8+...+2^k = 2^{k+1}-2, \quad \forall k \in \mathbb{Z},\ k \geq 1.$

When $n = k + 1$, RHS = $2^{k+1+1}-2=2^{k+2}-2$

$$\begin{aligned}\text{LHS} &= 2+4+8+...+2^k+2^{k+1} \\ &= 2^{k+1}-2+2^{k+1} \\ &= 2\left(2^{k+1}\right)-2 \\ &= 2^{k+2}-2 \\ &= \text{RHS}\end{aligned}$$

If the statement is true when $n=k$, it is also true for $n=k+1$.

Given that is true when n = 1, by induction $2+4+8+...+2^n = 2^{n+1}-2, \quad \forall n \in \mathbb{Z},\ n \geq 1.$

Question 4

RTP: $1^3 + 2^3 + 3^3 + ...n^3 = \frac{n^2}{4}(n+1)^2, \qquad \forall n \in \mathbb{Z},\ n \geq 1.$

When $n = 1$:

LHS = 1^3=1

RHS= $\frac{1^2}{4}(1+1)^2 = 1$

The initial case is true.

Assume the statement is true for $n = k.$

i.e. $\quad 1^3 + 2^3 + 3^3 + ...k^3 = \frac{k^2}{4}(k+1)^2, \qquad \forall k \in \mathbb{Z},\ k \geq 1.$

When $n = k+1$, RHS= $\frac{(k+1)^2}{4}(k+2)^2$

$$
\begin{aligned}
\text{LHS} &= 1^3 + 2^3 + 3^3 + ... + k^3 + (k+1)^3 \\
&= \frac{k^2}{4}(k+1)^2 + (k+1)^3 \\
&= \frac{(k+1)^2}{4}\left(k^2 + 4(k+1)\right) \\
&= \frac{(k+1)^2}{4}\left(k^2 + 4k + 4\right) \\
&= \frac{(k+1)^2}{4}(k+2)^2 \\
&= \text{RHS}
\end{aligned}
$$

If the statement is true when $n = k,$ it is also true for $n = k+1.$

Given that is true when n = 1, by induction $1^3 + 2^3 + 3^3 + ...n^3 = \frac{n^2}{4}(n+1)^2, \qquad \forall n \in \mathbb{Z},\ n \geq 1.$

Question 5

a RTP: $1+3+5+...+(2n-1)=n^2, \quad \forall n \in \mathbb{Z},\ n \geq 1.$

When n = 1:

LHS = 1^3=1

RHS=$1^2=1$

The initial case is true.

Assume the statement is true for $n=k.$

i.e. $1+3+5+...+(2k-1)=k^2, \quad \forall k \in \mathbb{Z},\ k \geq 1.$

When $n=k+1,$ RHS=$(k+1)^2$

$$\begin{aligned}\text{LHS} &= 1+3+5+...+2k-1+\left(2(k+1)-1\right)\\ &= k^2+(2k+1)\\ &= (k+1)^2\\ &= \text{RHS}\end{aligned}$$

b If the statement is true when $n=k,$ it is also true for $n=k+1.$

Given that is true when n = 1, by induction $1+3+5+...+(2n-1)=n^2, \quad \forall n \in \mathbb{Z},\ n \geq 1.$

Question 6

RTP: $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+...+\frac{1}{2^n}=\frac{2^n-1}{2^n}, \qquad \forall n\in\mathbb{Z},\ n\geq 1.$

When n = 1:

LHS $=\frac{1}{2}$

RHS$=\frac{2^1-1}{2^1}=\frac{1}{2}$

The initial case is true.

Assume the statement is true for $n=k.$

i.e. $\quad +\frac{1}{2^2}+\frac{1}{2^3}+...+\frac{1}{2^k}=\frac{2^k-1}{2^k}, \qquad \forall k\in\mathbb{Z},\ k\geq 1.$

When $n=k+1$, RHS$=\frac{2^{k+1}-1}{2^{k+1}}$

$$\begin{aligned}\text{LHS}&=\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+...+\frac{1}{2^k}+\frac{1}{2^{k+1}}\\&=\frac{2^k-1}{2^k}+\frac{1}{2^{k+1}}\\&=\frac{2\left(2^k-1\right)+1}{2^{k+1}}\\&=\frac{2^{k+1}-2+1}{2^{k+1}}\\&=\frac{2^{k+1}-1}{2^{k+1}}\\&=\text{RHS}\end{aligned}$$

If the statement is true when $n=k,$ it is also true for $n=k+1.$

Given that is true when n = 1, by induction $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+...+\frac{1}{2^n}=\frac{2^n-1}{2^n}, \qquad \forall n\in\mathbb{Z},\ n\geq 1.$

Question 7

RTP: $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+...+\frac{1}{n\left(n+1\right)}=\frac{n}{n+1}, \qquad \forall n\in\mathbb{Z},\ n\geq 1.$

When n = 1:

LHS $=\frac{1}{2}$

RHS$=\frac{1}{1+1}=\frac{1}{2}$

The initial case is true.

Assume the statement is true for $n=k.$

i.e. $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+...+\frac{1}{k\left(k+1\right)}=\frac{k}{k+1}, \qquad \forall k\in\mathbb{Z},\ k\geq 1.$

When $n=k+1,$ RHS$=\frac{k+1}{k+2}$

$$\begin{aligned}
\text{LHS} &= \frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+...+\frac{1}{k\left(k+1\right)}+\frac{1}{\left(k+1\right)\left(k+2\right)} \\
&= \frac{k}{k+1}+\frac{1}{\left(k+1\right)\left(k+2\right)} \\
&= \frac{k(k+2)+1}{\left(k+1\right)\left(k+2\right)} \\
&= \frac{k^2+2k+1}{\left(k+1\right)\left(k+2\right)} \\
&= \frac{\left(k+1\right)^2}{\left(k+1\right)\left(k+2\right)} \\
&= \frac{\left(k+1\right)}{\left(k+2\right)} \\
&= \text{RHS}
\end{aligned}$$

If the statement is true when $n=k,$ it is also true for $n=k+1.$ Given that is true when n=1, by induction, $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+...+\frac{1}{n\left(n+1\right)}=\frac{n}{n+1}, \qquad \forall n\in\mathbb{Z},\ n\geq 1.$

Question 8

RTP: $1\times3\times5+2\times4\times6+...+n(n+2)(n+4)=\dfrac{n(n+1)(n+4)(n+5)}{4} \qquad \forall n\in\mathbb{Z},\ n\geq1.$

When n = 1:

LHS $=15$

RHS$=\dfrac{1\times2\times5\times6}{4}=15$

The initial case is true.

Assume the statement is true for $n=k$.

i.e. $1\times3\times5+2\times4\times6+...+k(k+2)(k+4)=\dfrac{k(k+1)(k+4)(k+5)}{4} \qquad \forall k\in\mathbb{Z},\ k\geq1.$

When $n=k+1$, RHS $=\dfrac{(k+1)(k+2)(k+5)(k+6)}{4} \qquad \forall k\in\mathbb{Z},\ k\geq1.$

$$
\begin{aligned}
\text{LHS} &= 1\times3\times5+2\times4\times6+...+(k)(k+2)(k+4)+(k+1)(k+3)(k+5) \\
&= \frac{k(k+1)(k+4)(k+5)}{4}+(k+1)(k+3)(k+5) \\
&= \frac{k(k+1)(k+4)(k+5)+4(k+1)(k+3)(k+5)}{4} \\
&= \frac{(k+1)(k+5)[k(k+4)+4(k+3)]}{4} \\
&= \frac{(k+1)(k+5)[k^2+4k+4k+12]}{4} \\
&= \frac{(k+1)(k+5)[k^2+8k+12]}{4} \\
&= \frac{(k+1)(k+5)(k+2)(k+6)}{4} \\
&= \frac{(k+1)(k+2)(k+5)(k+6)}{4} \\
&= \text{RHS}
\end{aligned}
$$

If the statement is true when $n=k$, it is also true for $n=k+1$.

Given that is true when n = 1, by induction
$1\times3\times5+2\times4\times6+...+n(n+2)(n+4)=\dfrac{n(n+1)(n+4)(n+5)}{4} \qquad \forall n\in\mathbb{Z},\ n\geq1.$

Question 9

RTP: $\left(x-1\right)$ is a factor of $x^n-1, \quad \forall n \in \mathbb{Z},\ n \geq 0.$

When n = 1:

LHS $x-1$ is $x^1-1.$

The initial case is true.

Assume the statement is true for $n=k.$

i.e. $\quad \left(x^k-1\right)=a(x-1), \quad \forall k \in \mathbb{Z},\ n \geq 0.$

When $n=k+1,$

$$
\begin{aligned}
&x^{k+1}-1 \\
&= x^k.x-1 \\
&= x\left(x^k-1\right)+x-1 \\
&= ax\left(x-1\right)+\left(x-1\right) \qquad\qquad *x^k-1=a\left(x-1\right) \\
&= \left(x-1\right)\left(ax+1\right)
\end{aligned}
$$

If the statement is true when $n=k,$ it is also true for $n=k+1.$

Given that is true when n = 1, by induction $\left(x-1\right)$ is a factor of $x^n-1, \quad \forall n \in \mathbb{Z},\ n \geq 0.$

Question 10

RTP: $1\times 2\times 3\times ...\times n \geq 3^n, \qquad \forall n\in\mathbb{Z},\ n>6.$

When $n = 6$:

$$\begin{aligned}&\text{LHS} = 1\times 2\times 3\times 4\times 5\times 6\times 7 = 7! = 5040\\&\text{RHS} = 3^7 = 2187\\&5040 > 2187\end{aligned}$$

The initial case is true.

Assume the statement is true for $n = k$.

i.e. $\quad 1\times 2\times 3\times ...\times k \geq 3^k, \qquad \forall k\in\mathbb{Z},\ n>6.$

$$\begin{aligned}&\text{If } k! \geq 3^k\\&\text{then } 3k! \geq 3^k.3\\&\text{i.e. } 3k! \geq 3^{k+1}\end{aligned}$$

$$\begin{aligned}&\text{Similarly} \quad k!\left(k+1\right) \geq 3^k\left(k+1\right)\\&\text{and as } k+1>3,\ k!(k+1)\geq 3k!\geq 3^{k+1}\end{aligned}$$

$$\text{hence } k!\left(k+1\right)\geq 3^{k+1}$$

Alternatively

$3^{k+1} < 3^k (k + 1)$ as it requires multiplication by a number greater than one for

$3^k\left(k+1\right)=3^{k+1}\dfrac{\left(k+1\right)}{3}$ to be true.

When $n = k+1$

$$1\times 2\times 3\times ...\times k\times\left(k+1\right)\geq 3^k\left(k+1\right)$$

$$\begin{aligned}3^k\left(k+1\right) &= 3^k\times 3\times\frac{\left(k+1\right)}{3}\\&= 3^{k+1}\frac{\left(k+1\right)}{3}\end{aligned}$$

When $\quad n>6,$

$$\frac{k+1}{3}>1$$

$$\therefore 3^k\left(k+1\right)>3^{k+1}$$

If the statement is true when $n = k$, it is also true for $n = k+1$.

Given that is true when $n > 6$, by induction $1\times 2\times 3\times ...\times n \geq 3^n$, $\forall n \in \mathbb{Z},\ n > 6$.

Question 11

RTP: $7^n + 2\times 13^n$ is a multiple of three $\forall n \in \mathbb{Z},\ n \geq 0$.

When $n = 0$:

LHS $1 + 2\times 1 = 3$

The initial case is true.

Assume the statement is true for $n = k$.

i.e. $7^k + 2\times 13^k$ is a multiple of three $\forall k \in \mathbb{Z},\ k \geq 0$.

If $7^k + 2\times 13^k$ is a multiple of three ,
we can write $7^k + 2\times 13^k = 3M, M \in \mathbb{Z}$

When $n = k+1$

$$
\begin{aligned}
&7^{k+1} + 2\times 13^{k+1} \\
&= 7^k.7 + 2\times 13^k.13 \\
&= 7^k.7 + 2.13^k.(7+6) \\
&= 7^k.7 + 14.13^k + 12.13^k \\
&= 7\left(7^k + 2.13^k\right) + 12.13^k \\
&= 7.3M + 12.13^k \\
&= 3\left(7M + 4.13^k\right) \text{ which is a multiple of three.}
\end{aligned}
$$

If the statement is true when $n = k$, it is also true for $n = k+1$.

Given that is true when $n = 0$, by induction $7^n + 2\times 13^n$ is a multiple of three $\forall n \in \mathbb{Z},\ n \geq 0$.

An alternative approach:

If $7^k + 2\times 13^k$ is a multiple of three ,

we can write $7^k + 2\times 13^k = 3M, M \in \mathbb{Z}$

$\Rightarrow 7^k = 3M - 2.13^k$

When $n = k+1$

$7^{k+1} + 2\times 13^{k+1}$

$= 7^k.7 + 2\times 13^{k+1}$

$= 7\left(3M - 2.13^k\right) + 2\times 13^k.13$

$= 21M - 14.13^k + 26.13^k$

$= 21M + 12.13^k$

$= 3\left(7M + 4.13^k\right)$ which is a multiple of three.

Question 12

RTP: $2-4+8+...+(-1)^{n+1}2^n = \frac{2}{3}\left(1+(-1)^{n+1}2^n\right), \quad \forall n \in \mathbb{Z},\ n \geq 1.$

When n = 1:

LHS = $\left(-1\right)^{1+1}2^1 = 2$

RHS $= \frac{2}{3}\left(1+(-1)^{1+1}2^1\right) = 2$

The initial case is true.

Assume the statement is true for $n = k.$

$$2-4+8+...+(-1)^{k+1}2^k = \frac{2}{3}\left(1+(-1)^{k+1}2^k\right), \quad \forall k \in \mathbb{Z},\ k \geq 1.$$

When $n = k+1$

$$\text{RHS} = \frac{2}{3}\left(1+(-1)^{k+2}2^{k+1}\right)$$

$$\begin{aligned}
\text{LHS} &= 2-4+8.....+\left(-1\right)^{k+1}2^k+\left(-1\right)^{k+2}2^{k+1} \\
&= \frac{2}{3}\left(1+(-1)^{k+1}2^k\right)+\left(-1\right)^{k+2}2^{k+1} \\
&= \frac{2}{3}\left[1+(-1)^{k+1}2^k+\frac{3}{2}\left(-1\right)^{k+2}2^{k+1}\right] \\
&= \frac{2}{3}\left[1+\frac{(-1)^{k+1}2^k(-1)\ 2}{(-1)\ 2}+\frac{3}{2}\left(-1\right)^{k+2}2^{k+1}\right] \\
&= \frac{2}{3}\left[1-\frac{(-1)^{k+2}2^{k+1}}{2}+\frac{3}{2}\left(-1\right)^{k+2}2^{k+1}\right] \\
&= \frac{2}{3}\left[1+\left(-1\right)^{k+2}2^{k+1}\right] \\
&= \text{RHS}
\end{aligned}$$

If the statement is true when $n = k,$ it is also true for $n = k+1.$

Given that is true when n = 1, by induction $2-4+8+...+(-1)^{n+1}2^n = \frac{2}{3}\left(1+(-1)^{n+1}2^n\right), \quad \forall n \in \mathbb{Z},\ n \geq 1.$

Question 1

a Cannot be determined – number of columns in matrix 1 does not equal the number of rows in matrix 2.

b $\begin{bmatrix} -1 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}$

c $\begin{bmatrix} -1 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -3 & 3 & -3 \\ -3 & 3 & -3 \end{bmatrix}$

d Cannot be determined – number of columns in matrix 1 does not equal the number of rows in matrix 2.

e $\begin{bmatrix} -1 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 6 & 5 & 4 \end{bmatrix}$

Question 2

$$A^{-1} = \frac{1}{5}\begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 13 \\ -4 \end{bmatrix}$$

$$A^{-1}AB = A^{-1}\begin{bmatrix} 13 \\ -4 \end{bmatrix}$$

$$B = \frac{1}{5}\begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 13 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$AC = \begin{bmatrix} 13 \\ 6 \end{bmatrix}$$

$$A^{-1}AC = A^{-1}\begin{bmatrix} 13 \\ 6 \end{bmatrix}$$

$$C = \frac{1}{5}\begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 13 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

$$DA = \begin{bmatrix} 6 & 19 \end{bmatrix}$$

$$DAA^{-1} = \begin{bmatrix} 6 & 19 \end{bmatrix}A^{-1}$$

$$= \frac{1}{5}\begin{bmatrix} 6 & 19 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 \end{bmatrix}$$

$$EA = \begin{bmatrix} 5 & 0 \end{bmatrix}$$

$$EAA^{-1} = \begin{bmatrix} 5 & 0 \end{bmatrix}A^{-1}$$

$$= \frac{1}{5}\begin{bmatrix} 5 & 0 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 \end{bmatrix}$$

Question 3

$$AC = B$$

$$C = A^{-1}B$$

$$A^{-1} = \frac{1}{11}\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}$$

$$C = \frac{1}{11}\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 4 & 21 \\ 9 & 17 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 \\ 2 & 5 \end{bmatrix}$$

Question 4

a Given $X_{2\times2}$, $Y_{2\times1}$ and $Z_{1\times2}$ the only possibilities are XY and ZX.

b ZX

c

$$ZX = \begin{bmatrix} 210 & 120 \end{bmatrix}\begin{bmatrix} 75 & 25 \\ 20 & 80 \end{bmatrix}$$

$$= \begin{bmatrix} 210\times75+120\times20 & 210\times25+120\times80 \end{bmatrix}$$

$$= \begin{bmatrix} 18150 & 14850 \end{bmatrix}$$

18 150 Australian Stamps and 14 850 stamps from the Rest of the World required to fill these requests.

Question 5

$$A^2 = \begin{bmatrix} x & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x & 1 \\ 0 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} x^2 & x+3 \\ 0 & 9 \end{bmatrix}$$
$$A^2 + A = \begin{bmatrix} x^2 & x+3 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} x & 1 \\ 0 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} x^2+x & x+4 \\ 0 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 6 & x^2-8 \\ p & q \end{bmatrix} = \begin{bmatrix} x^2+x & x+4 \\ 0 & 12 \end{bmatrix}$$

$$x^2 + x = 6 \qquad\qquad x^2 - 8 = x + 4$$
$$x^2 + x - 6 = 0 \qquad\qquad x^2 - x - 12 = 0$$
$$(x+3)(x-2) = 0 \qquad\qquad (x-4)(x+3) = 0$$
$$x = -3, 2 \qquad\qquad x = -3, 4$$
$$\Rightarrow x = -3,\ p = 0,\ q = 12$$

Question 6

$$\text{RHS} = \frac{2\tan\theta}{\tan^2\theta + 1}$$
$$= \frac{2\tan\theta}{\sec^2\theta}$$
$$= 2 \times \frac{\sin\theta}{\cos\theta} \div \frac{1}{\cos^2\theta}$$
$$= 2 \times \frac{\sin\theta}{\cos\theta} \times \cos^2\theta$$
$$= 2\sin\theta\cos\theta$$
$$= \sin 2\theta$$
$$= \text{LHS}$$

Question 7

$$
\begin{aligned}
\text{LHS} &= \sin 5x \cos 3x - \cos 6x \sin 2x \\
&= \frac{1}{2}(\sin 8x + \sin 2x) - \frac{1}{2}(\sin 8x - \sin 4x) \\
&= \frac{1}{2}(\sin 8x + \sin 2x - \sin 8x + \sin 4x) \\
&= \frac{1}{2}(\sin 4x + \sin 2x) \\
&= \frac{1}{2}(2\sin 3x \cos x) \\
&= \sin 3x \cos x \\
&= \text{RHS}
\end{aligned}
$$

Question 8

a $R\cos(\theta+\alpha) = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

$5\cos\theta - 3\sin\theta = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

$R\cos\alpha = 5$ $\quad$ $R\sin\alpha = 3$

$\cos\alpha = \frac{5}{R}$ $\quad$ $\sin\alpha = \frac{3}{R}$

$\tan\alpha = \frac{3}{5}$

$\alpha = 0.54 \text{ radians}$

$$
\begin{aligned}
R &= \sqrt{5^2 + 3^2} \\
&= \sqrt{34}
\end{aligned}
$$

$5\cos\theta - 3\sin\theta = \sqrt{34}\cos(\theta + 0.54)$

b $\cos(\theta + 0.54)$ has a minimum value of -1

$\sqrt{34}\cos(\theta + 0.54)$ has a minimum value of -$\sqrt{34}$

$$
\begin{aligned}
\cos(\theta + 0.54) &= -1 \\
\theta + 0.54 &= \pi \\
\theta &= 2.60 \text{ radians}
\end{aligned}
$$

Question 9

$A_{3\times1} \quad B_{2\times3} \quad C_{1\times4}$

$B_{2\times3}A_{3\times1}C_{1\times4}$

BAC is the order of multiplication

$$\mathrm{BAC} = \begin{bmatrix} 10 & 0 & 10 & 10 \\ 8 & 0 & 8 & 8 \end{bmatrix}$$

Question 10

$$\begin{aligned} A^2 &= \begin{bmatrix} 2x & x \\ 4 & y \end{bmatrix}\begin{bmatrix} 2x & x \\ 4 & y \end{bmatrix} \\ &= \begin{bmatrix} 4x^2+4x & 2x^2+xy \\ 8x+4y & 4x+y^2 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} 4x^2+4x & 2x^2+xy \\ 8x+4y & 4x+y^2 \end{bmatrix} = \begin{bmatrix} 24 & p \\ 0 & q \end{bmatrix}$$

$$\begin{aligned} 4x^2+4x &= 24 \\ 4x^2+4x-24 &= 0 \\ x^2+x-6 &= 0 \\ (x+3)(x-2) &= 0 \\ x &= -3, 2 \end{aligned}$$

If $x = 2$

$$\begin{aligned} 8x+4y &= 0 \\ 16+4y &= 0 \\ y &= -4 \end{aligned}$$

If $x = -3$

$$\begin{aligned} 8x+4y &= 0 \\ -24+4y &= 0 \\ y &= 6 \end{aligned}$$

$$\begin{aligned} p &= 2x^2+xy \\ &= 2(2)^2+2(-4) \\ &= 0 \\ q &= 4x+y^2 \\ &= 4(2)+(-4)^2 \\ &= 24 \end{aligned}$$

$$\begin{aligned} p &= 2x^2+xy \\ &= 2(-3)^2+(-3)\times6 \\ &= 0 \\ q &= 4x+y^2 \\ &= 4(-3)+6^2 \\ &= 24 \end{aligned}$$

Question 11

There is no conflict as the argument is correct provided A^{-1} exists.

A^{-1} only exists if A is a square matrix.

Question 12

$A_{2\times3}$ $B_{1\times3}$ $C_{3\times1}$ $D_{3\times3}$

~~AA~~ ~~BA~~ ~~CA~~ ~~DA~~

~~AB~~ ~~BB~~ CB ~~DB~~

AC BC ~~CC~~ DC

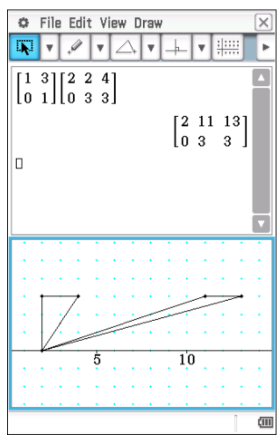
AD BD ~~CD~~ DD

AC, AD, BC, BD, CB, DC and DD are the only possible products.

Question 13

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 & 4 \\ 0 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 11 & 13 \\ 0 & 3 & 3 \end{bmatrix}$$

$A'(2, 0)$ $B'(11, 3)$ $C'(13, 3)$



Question 14

$$\begin{aligned}\text{LHS} &= \sec x \ \operatorname{cosec} x \ \ \cot x \\ &= \frac{1}{\cos x}\times\frac{1}{\sin x}\times\frac{\cos x}{\sin x} \\ &= \frac{1}{\sin^2 x} \\ &= \operatorname{cosec}^2 x \\ &= 1+\cot^2 x \\ &= \text{RHS}\end{aligned}$$

Question 15

$$R\sin(\theta+\alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$$

$$7\sin x+\cos x = R\sin x\cos\alpha + R\cos x\sin\alpha$$

$$R\cos\alpha = 7 \qquad\qquad R\sin\alpha = 1$$

$$\cos\alpha = \frac{7}{R} \qquad\qquad \sin\alpha = \frac{1}{R}$$

$$\tan\alpha = \frac{1}{7}$$

$$\alpha = 0.14$$

$$\begin{aligned}R &= \sqrt{7^2+1^2} \\ &= \sqrt{50} \\ &= 5\sqrt{2}\end{aligned}$$

$$7\sin x+\cos x = 5\sqrt{2}\sin(x+0.14) = 5$$

$$\sin(x+0.14) = \frac{1}{\sqrt{2}}$$

$$x+0.14 = \frac{\pi}{4}, \frac{3\pi}{4} + 2\pi n, n\in\mathbb{Z}$$

$$x = \begin{cases}0.64 \\ 2.21\end{cases} + 2\pi n, \ \ n\in\mathbb{Z}$$

Question 16

RTP: $12+19+31+...+(5(1+2^{n-1})+2n)=n(n+6)+5(2^n-1) \quad \forall n\in\mathbb{Z}, n\ge 1$

When $n=1$

$\text{LHS } =(5(1+2^0)+2(1))=12$
$\text{RHS}=1(1+6)+5(2^1-1)=12$

The statement is true for the initial case.

Assume the statement is true for $n=k$

$12+19+31+...+(5(1+2^{k-1})+2k)=k(k+6)+5(2^k-1) \quad \forall k\in\mathbb{Z}, k\ge 1$

When $n=k+1$

$$12+19+31+...+\left(5(1+2^{k-1})+2k\right)+\left(5(1+2^{k+1-1})+2(k+1)\right)$$
$$=k(k+6)+5(2^k-1)+\left(5(1+2^{k+1-1})+2(k+1)\right)$$
$$=k(k+6)+5.2^k-5+5+5.2^k+2k+2$$
$$=k(k+6)+2\times 5.2^k-5+5+2k+2$$
$$=k(k+6)+5.2^{k+1}-5+2k+7$$
$$=k^2+6k+5(2^{k+1}-1)+2k+7$$
$$=k^2+8k+7+5(2^{k+1}-1)$$
$$=(k+1)(k+7)+5(2^{k+1}-1)$$

If the statement is true for $n=k$ then it is also true for $n=k+1$. The statement is true for $n=1$ so by the principles of mathematical induction, the statement is true for $n\ge 1$.

Question 17

RTP: $3^{2n+4}-2^{2n}=5M, M\in\mathbb{Z}\ \ \forall n\in\mathbb{Z}, n\geq 1$

When $n=1$

$3^{2(1)+4}-2^{2(1)}$
$=3^6-2^2$
$=725$
725 is a multiple of 5 so the statement is true for the initial case.

Assume the statement is true for $n=k$ i.e.

$3^{2k+4}-2^{2k}=5M, M\in\mathbb{Z}\ \ \forall k\in\mathbb{Z}, k\geq 1$

When $n=k+1$

$3^{2(k+1)+4}-2^{2(k+1)}$
$=3^{2k+4+2}-2^{2k+2}$
$=3^2\times 3^{2k+4}-2^2\times 2^{2k}$
$=9\times 3^{2k+4}-4\times 2^{2k}$
$=5\times 3^{2k+4}+4\times 3^{2k+4}-4\times 2^{2k}$
$=5\times 3^{2k+4}+4(3^{2k+4}-2^{2k})$
$=5\times 3^{2k+4}+4\times 5M$
$=5(3^{2k+4}+4M)$ which is clearly a multiple of 5

If the statement is true for $n=k$ then it is also true for $n=k+1$. The statement is true for $n=1$ so by the principles of mathematical induction, the statement is true for all positive integer n.

Question 18

RTP: $5^n + 7\times 13^n = 8M, M \in \mathbb{Z} \quad \forall n \in \mathbb{Z}, n \geq 1$

When $n = 1$

$5^1 + 7\times 13^1 = 96$
96 is a multiple of 8 so the statement is true for the initial case.

Assume the statement is true for $n = k$ i.e.

$5^k + 7\times 13^k = 8M, M \in \mathbb{Z}, \forall k \in \mathbb{Z}, k \geq 1$

When $n = k+1$

$5^{k+1} + 7\times 13^{k+1}$
$= 5.5^k + 7\times 13^k.13$
$= 5\times 5^k + 5\times 7\times 13^k + 8\times 7\times 13^k$
$= 5(5^k + 7\times 13^k) + 8\times 7\times 13^k$
$= 5\times 8M + 8\times 7\times 13^k$
$= 8(5M + 7\times 13^k)$ which is a multiple of 8.

If the statement is true for $n = k$ then it is also true for $n = k+1$. The statement is true for $n = 1$ so by the principles of mathematical induction, the statement is true for all positive integer $n \geq 1$.

Question 19

RTP: $r + r^2 + r^3 + ...r^n = \dfrac{r(r^n - 1)}{r-1} \quad \forall n \in \mathbb{Z}, n \geq 1$

When $n = 1$

$\text{LHS} = r$

$\text{RHS} = \dfrac{r(r^1 - 1)}{r-1} = r$

The statement is true for the initial case.

Assume the statement is true for $n = k$ i.e.

$r + r^2 + r^3 + ...r^k = \dfrac{r(r^k - 1)}{r-1} \quad \forall k \in \mathbb{Z}, k \geq 1$

When $n = k+1$

$$
\begin{aligned}
& r + r^2 + r^3 + ...r^k + r^{k+1} \\
&= \frac{r(r^k - 1)}{r-1} + r^{k+1} \\
&= \frac{r(r^k - 1)}{r-1} + \frac{r^{k+1}(r-1)}{r-1} \\
&= \frac{r(r^k - 1) + r.r^k(r-1)}{r-1} \\
&= \frac{r(r^k - 1 + r^k(r-1))}{r-1} \\
&= \frac{r(r^k + r^k(r-1) - 1)}{r-1} \\
&= \frac{r(r^k(1 + r - 1) - 1)}{r-1} \\
&= \frac{r(r^k.r - 1)}{r-1} \\
&= \frac{r(r^{k+1} - 1)}{r-1}
\end{aligned}
$$

If the statement is true for $n = k$ then it is also true for $n = k+1$. The statement is true for $n = 1$ so by the principles of mathematical induction, the statement is true for all positive integer $n \geq 1$.